FIRST ORDER

- Separation of variables $\frac{dy}{dx} = f(x)\,g(y)$ $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

- Integrating factor $\frac{dy}{dx} + g(x)\,y = h(x)$ $p(x) = e^{\int g(x)\,dx}$

$$\frac{d}{dx}\left(p_x y\right) = h(x)p(x) \quad y(x)p(x) = C + \int dx\, p(x)\,h(x)$$

- Homogeneous when $\frac{dy}{dx} = F(y/x)$ Set $v = y/x$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

substitute and solve for v, then for y.

- Bernoulli's equation $\frac{dy}{dx} + y\,P(x) = y^n\,Q(x)$ Set $z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$$

Substitute and solve for z, then for y.

- Ricatti's equation $\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$

  — Set $y = \frac{-u'}{uR}$ gives linear 2^{nd} order $\frac{d^2u}{dx^2} - \left(Q + \frac{R'}{R}\right)\frac{du}{dx} + PRu = 0$

  solve for u, replace y.

  — One integral is known $v(x)$ Set $y = v + \frac{1}{z}$

  gives $z' + (Q + 2Rv)z = -R$ linear 1^{st} order

  — Two integrals are known $v_1(x)$ $v_2(x)$

  gives $\ln\left(\frac{y - v_1}{y - v_2}\right) = \int dx\,(v_1 - v_2)\,R$

- Lienard's equation $\frac{d^2x}{dt^2} + v\,f(x)\frac{dx}{dt} + g(x) = 0$

Set $y = \frac{dx}{dt}$ and substitute $\begin{cases} \frac{dy}{dt} = -v\,f(x)\,y - g(x) \\ \frac{dx}{dt} = y \end{cases}$

Clairaut's equation

$$y = px + f(p) \qquad p = \frac{dy}{dx}$$

$$p = c \Rightarrow y = cx + f(c)$$

OR

$$x = -f'(p) \Rightarrow y = P(x)$$

Variation of parameter

$y = Ae^{\mu x}$ general solution

Set $y = A(x)e^{\mu x}$ and substitute in the equation

SECOND-ORDER

Euler equation

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = 0$$

set $x = e^t$ $\rightarrow$ $x\frac{dy}{dx} = \frac{dy}{dt}$

$(t = \ln x)$

and

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

substitute and solve for $y(t)$ then for $y(x)$