Exercise 1.7

A smooth wire has the shape of a cycloid given parametrically by $x = a(\phi + \sin\phi)$, $y = a(1 - \cos\phi)$, $(-\pi < \phi < \pi)$. A bead is released from rest where $\phi = \phi_0$. Using conservation of energy confirm that

$$a\dot{\phi}^2 \cos^2 \tfrac{1}{2}\phi = g(\sin^2 \tfrac{1}{2}\phi_0 - \sin^2 \tfrac{1}{2}\phi).$$

Hence show that the period of oscillation of the bead is $4\pi\sqrt{(a/g)}$ (i.e., independent of ϕ_0). This is known as the **tautochrone**.

$$x = a(\phi + \sin\phi)$$
$$y = a(1 - \cos\phi)$$

$$\frac{1}{2}mv^2 + mgy = mgy_0$$

$$\frac{1}{2}v^2 + ga(1-\cos\phi) = ga(1-\cos\phi_0)$$

$$v^2 = 2ga(\cos\phi - \cos\phi_0)$$

$$\frac{dx}{d\phi} = a(1+\cos\phi) \qquad \frac{dy}{d\phi} = a(\sin\phi)$$

$$v = \frac{ds}{dt} = \frac{\sqrt{dx^2 + dy^2}}{dt} = \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2} \; \frac{d\phi}{dt}$$

$$v^2 = \left(a^2(1+\cos\phi)^2 + a^2\sin^2\phi\right)\dot{\phi}^2$$

$$a^2(2 + 2\cos\phi)\dot{\phi}^2 = 2ga(\cos\phi - \cos\phi_0)$$

$$a(1+\cos\phi)\dot{\phi}^2 = g(\cos\phi - \cos\phi_0)$$

$$a\left(2\cos^2\frac{\phi}{2}\right)\dot{\phi}^2 = g\left(1 - 2\sin^2\frac{\phi}{2} - \left(1 - 2\sin^2\frac{\phi_0}{2}\right)\right)$$

$$a\dot{\phi}^2\cos^2\frac{\phi}{2} = g\left(\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}\right)$$

$$T = \sqrt{\frac{a}{g}} \int_0^{\emptyset_0} \frac{\cos\frac{\emptyset}{2}}{\sqrt{\sin^2\frac{\emptyset_0}{2} - \sin^2\frac{\emptyset}{2}}} d\emptyset$$

$$= \sqrt{\frac{a}{g}} \int_0^{\emptyset_0} \frac{\cos\frac{\emptyset}{2}}{\sin\frac{\emptyset_0}{2}\sqrt{\left(1 - \frac{\sin^2\frac{\emptyset}{2}}{\sin^2\frac{\emptyset_0}{2}}\right)}} d\emptyset$$

Let $u = \frac{\sin\frac{\emptyset}{2}}{\sin\frac{\emptyset_0}{2}}$ so that $du = \frac{\cos\frac{\emptyset}{2}}{2\sin\frac{\emptyset_0}{2}} d\emptyset$ and $d\emptyset = \frac{2\sin\frac{\emptyset_0}{2}}{\cos\frac{\emptyset}{2}} du$ and the integral becomes

$$T = \sqrt{\frac{a}{g}} \int_{\emptyset=0}^{\emptyset=\emptyset_0} \frac{2}{\sqrt{1-u^2}} du$$

$$= \sqrt{\frac{a}{g}} \left[2\arcsin\left(\frac{\sin\frac{\emptyset}{2}}{\sin\frac{\emptyset_0}{2}}\right)\right]_0^{\emptyset_0}$$

$$= \pi\sqrt{\frac{a}{g}}$$

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