Nonlinear Ordinary Differential Equations, Jordan and Smith

Exercise 1.5

> Show that every phase path of
>
> $$\ddot{x} + \varepsilon|x|\operatorname{sgn}\dot{x} + x = 0, \quad 0 < \varepsilon < 1,$$
>
> is an isochronous spiral (that is, every circuit of the origin on every path occur in the same time).

If a phase path has equation $y = f(x)$ then a similar phase path has equation $y = kf\left(\frac{x}{k}\right)$ and the transit times between corresponding pairs of points are $T_1 = \int_a^b \frac{dx}{f(x)}$ and $T_2 = \int_{ka}^{kb} \frac{dx}{kf\left(\frac{x}{k}\right)}$.

Making the substitution $u = \frac{x}{k}$ the integral for T_2 becomes $T_2 = \int_a^b \frac{kdu}{kf(u)} = \int_a^b \frac{du}{f(u)} = T_1$.

In the first and third quadrants the given equation becomes $\ddot{x} = -x(1+\varepsilon)$ whilst in the second and fourth quadrants the equation is $\ddot{x} = -x(1-\varepsilon)$.

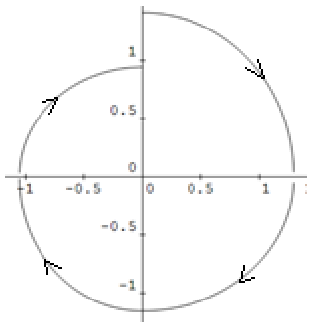
The resulting phase paths are made up of sections of ellipses with equations of the form $y^2 = C - x^2(1+\varepsilon)$ and $y^2 = C - x^2(1-\varepsilon)$ with those sections of ellipses in opposite quadrants being similar.

Since similar phase paths have the same transit time every circuit of the origin, on every path, occurs in the same time.

If a phase path passes through the point $(0, \sqrt{a})$ then the equation of the path in the first quadrant is $y = \sqrt{a - x^2(1+\varepsilon)}$, the equation of the path in the fourth quadrant is

$y = -\sqrt{\frac{a(1-\varepsilon)}{1+\varepsilon} - x^2(1-\varepsilon)}$, in the third quadrant the equation is $y = -\sqrt{\frac{a(1-\varepsilon)}{1+\varepsilon} - x^2(1+\varepsilon)}$ and in the second quadrant the equation is $y = \sqrt{\frac{a(1-\varepsilon)^2}{(1+\varepsilon)^2} - x^2(1-\varepsilon)}$.

On its return to the y axis the path reaches the point $\left(0, \frac{1-\varepsilon}{1+\varepsilon}\sqrt{a}\right)$ and since $\frac{1-\varepsilon}{1+\varepsilon} < 1$ the path is a spiral which approaches the origin.



An example showing part of a path with $a = 2$ and $\varepsilon = 0.2$.

The time required for one circuit can be found by evaluating integrals for parts of representative ellipses as follows.

$$2\int_0^{\sqrt{\frac{a}{1+\varepsilon}}} (a-(1+\varepsilon)x^2)^{-1/2}\,dx + 2\int_0^{\sqrt{\frac{a}{1-\varepsilon}}} (a-(1-\varepsilon)x^2)^{-1/2}\,dx$$

Using the standard integral, $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$, the time required for one circuit is found to be

$$\frac{2}{\sqrt{1+\varepsilon}}\arcsin\left(\sqrt{\frac{1+\varepsilon}{a}}\sqrt{\frac{a}{1+\varepsilon}}\right) + \frac{2}{\sqrt{1-\varepsilon}}\arcsin\left(\sqrt{\frac{1-\varepsilon}{a}}\sqrt{\frac{a}{1-\varepsilon}}\right) = \frac{\pi}{\sqrt{1+\varepsilon}} + \frac{\pi}{\sqrt{1-\varepsilon}}.$$

Bury Maths Tutor