A random variable X has probability distribution given by:

$$P(X = x) = \begin{cases} \dfrac{2x}{n(n+1)} & \text{for } x = 1,2,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

where n is a positive integer.

Prove that $Var(X) = \frac{(n+2)(n-1)}{18}$.

$$E(X) = \sum_{x=1}^{n} xP(X = x) = \sum_{x=1}^{n} \frac{2x^2}{n(n+1)} = \frac{2}{n(n+1)} \sum_{x=1}^{n} x^2$$

$$= \frac{2}{n(n+1)} \times \frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{3}$$

$$E(X)^2 = \frac{(2n+1)^2}{9}$$

$$E(X^2) = \sum_{x=1}^{n} x^2 P(X = x) = \sum_{x=1}^{n} \frac{2x^3}{n(n+1)} = \frac{2}{n(n+1)} \sum_{x=1}^{n} x^3$$

$$= \frac{2}{n(n+1)} \times \left(\frac{n(n+1)}{2}\right)^2 = \frac{n(n+1)}{2}$$

$$Var(X) = \frac{n(n+1)}{2} - \frac{(2n+1)^2}{9}$$

$$= \frac{9n(n+1)}{18} - \frac{2(2n+1)^2}{18}$$

$$= \frac{9n^2 + 9n - 8n^2 - 8n - 2}{18}$$

$$= \frac{n^2 + n - 2}{18}$$

$$= \frac{(n+2)(n-1)}{18}$$