8.

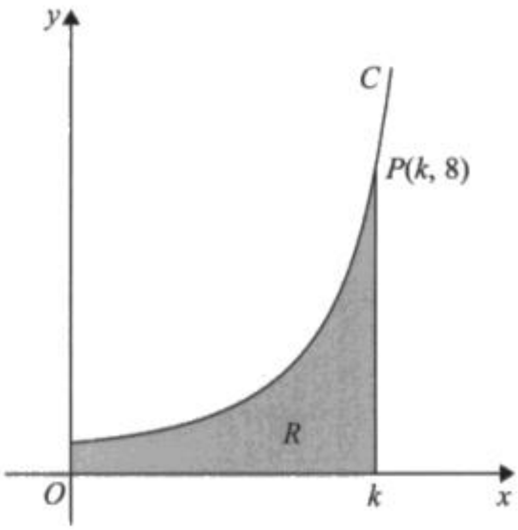

Diagram not drawn to scale

Figure 4

Figure 4 shows a sketch of part of the curve C with parametric equations

$$x = 3\theta \sin\theta, \qquad y = \sec^3\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point $P(k, 8)$ lies on C, where k is a constant.

(a) Find the exact value of k.

(2)

The finite region R, shown shaded in Figure 4, is bounded by the curve C, the y-axis, the x-axis and the line with equation $x = k$.

(b) Show that the area of R can be expressed in the form

$$\lambda \int_{\alpha}^{\beta} \left(\theta \sec^2\theta + \tan\theta \sec^2\theta\right) \mathrm{d}\theta$$

where λ, α and β are constants to be determined.

(4)

(c) Hence use integration to find the exact value of the area of R.

(6)

(a) $y = 8 \Rightarrow \sec^3\theta = 8 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{3} \Rightarrow x = 3 \times \frac{\pi}{3} \times \sin\frac{\pi}{3} = \frac{\sqrt{3}\pi}{2}$

(b) $\frac{dx}{d\theta} = 3\sin\theta + 3\theta\cos\theta \Rightarrow dx = (3\sin\theta + 3\theta\cos\theta)d\theta$

Area $= \int_0^{\frac{\sqrt{3}\pi}{2}} y\,dx = \int_0^{\frac{\pi}{3}} \sec^3\theta\,(3\sin\theta + 3\theta\cos\theta)d\theta = 3\int_0^{\frac{\pi}{3}} \left(\frac{\sin\theta}{\cos^3\theta} + \theta\sec^2\theta\right)d\theta$

$= 3\int_0^{\frac{\pi}{3}} (\sec^2\theta\tan\theta + \theta\sec^2\theta)\,d\theta$

$\lambda = 3 \quad \alpha = 0 \quad \beta = \frac{\pi}{3}$

(c) $3\int_0^{\frac{\pi}{3}} (\sec^2\theta\tan\theta + \theta\sec^2\theta)\,d\theta$

$= 3\int_0^{\frac{\pi}{3}} (\sec^2\theta\tan\theta)\,d\theta + 3\int_0^{\frac{\pi}{3}} (\theta\sec^2\theta)\,d\theta$

The first term above may be integrated by recognition or by substitution.

Using substitution:

Let $u = tan\,\theta$ then $\frac{du}{d\theta} = \sec^2\theta \Rightarrow d\theta = \frac{du}{\cos^2\theta}$

When $\theta = 0, u = 0$ and when $\theta = \frac{\pi}{3}, u = \sqrt{3}$.
The integral can now be written as $3\int_0^{\sqrt{3}} u\,du = \frac{9}{2}$.

Using integration by parts for the second term:

Let $u = \theta$ and $\frac{dv}{d\theta} = \sec^2\theta$
$\frac{du}{d\theta} = 1$ and $v = \tan\theta$

The integral can now be written as
$3[\theta\tan\theta]_0^{\frac{\pi}{3}} - 3\int_0^{\frac{\pi}{3}} \tan\theta\,d\theta = 3[\theta\tan\theta - \ln\sec\theta]_0^{\frac{\pi}{3}} = \sqrt{3}\pi - 3\ln\frac{1}{\cos\frac{\pi}{3}} = \sqrt{3}\pi - \ln 8$
Adding the two results gives $\frac{9}{2} + \sqrt{3}\pi - \ln 8$.