

A proof by induction

Show that $\frac{\prod_{r=1}^{4n} r!}{(2n)!}$ is square.

Proof by induction.

$$\text{Let } f(n) = \frac{\prod_{r=1}^{4n} r!}{(2n)!}$$

$$f(1) = \frac{1 \times 2 \times 6 \times 24}{2} = 144$$

$$\begin{aligned} f(k+1) &= f(k) \times \frac{(4k+1)!(4k+2)!(4k+3)!(4k+4)!}{(2k+1)(2k+2)} \\ &= f(k) \times \frac{((4k+1)!)^4 (4k+2)(4k+2)(4k+3)(4k+2)(4k+3)(4k+4)}{(2k+1)(2k+2)} \\ &= f(k) \times ((4k+1)!)^4 (2)(4k+2)(4k+3)(4k+2)(4k+3)(2) \\ &= f(k) \times \left[2((4k+1)!)^2 (4k+2)(4k+3) \right]^2 \end{aligned}$$

$f(1)$ is square.

If $f(k)$ is square then $f(k+1)$ is square.

By the principle of mathematical induction, $f(n)$ is square for all $n \in \mathbb{N}$.

Thanks to Don Berry for setting the problem.