$$w = f(x, y) \qquad x = e^r \cos\theta \qquad y = e^r \sin\theta$$

Show that $w_{xx} + w_{yy} = e^{-2r}(w_{rr} + w_{\theta\theta})$

$$\frac{\partial x}{\partial \theta} = -y \qquad \frac{\partial y}{\partial \theta} = x \qquad \frac{\partial x}{\partial r} = x \qquad \frac{\partial y}{\partial r} = y$$

$$w_r = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = x w_x + y w_y$$

$$w_{rr} = \frac{\partial x}{\partial r} w_x + x \frac{\partial}{\partial r}(w_x) + \frac{\partial y}{\partial r} w_y + y \frac{\partial}{\partial r}(w_y)$$

$$w_{rr} = x w_x + y w_y + x\left[w_{xx} x + w_{xy} y\right] + y\left[w_{yx} x + w_{yy} y\right]$$

$$\boxed{w_{rr} = w_r + x^2 w_{xx} + 2xy w_{xy} + y^2 w_{yy}}$$

$$w_\theta = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = -y w_x + x w_y$$

$$w_{\theta\theta} = -\frac{\partial y}{\partial \theta} w_x - y \frac{\partial}{\partial \theta}(w_x) + \frac{\partial x}{\partial \theta} w_y + x \frac{\partial}{\partial \theta}(w_y)$$

$$w_{\theta\theta} = -x w_x - y w_y - y\left[w_{xx}(-y) + w_{xy} x\right] + x\left[w_{yx}(-y) + w_{yy} x\right]$$

$$\boxed{w_{\theta\theta} = -w_r + y^2 w_{xx} - 2xy w_{xy} + x^2 w_{yy}}$$

$$w_{rr} + w_{\theta\theta} = (x^2 + y^2)(w_{xx} + w_{yy})$$

$$w_{rr} + w_{\theta\theta} = e^{2r}(w_{xx} + w_{yy})$$

$$w_{xx} + w_{yy} = e^{-2r}(w_{rr} + w_{\theta\theta})$$