$$z = y\phi(x^2 - y^2)$$

$$\frac{\partial z}{\partial x} = 2xy\,\phi'(x^2 - y^2)$$

$$\frac{\partial z}{\partial y} = \phi(x^2-y^2) - 2y^2\phi'(x^2-y^2)$$

$$\frac{z}{y^2} = \frac{\phi(x^2-y^2)}{y}$$

$$\frac{1}{x}\frac{\partial z}{\partial x} = 2y\,\phi'(x^2-y^2)$$

$$\frac{1}{y}\frac{\partial z}{\partial y} = \frac{\phi(x^2-y^2)}{y} - 2y\,\phi'(x^2-y^2)$$

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}$$

Show that $z = e^{y}\phi\left(y e^{\frac{x^2}{2y^2}}\right)$

satisfies $(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz$

let $u = \frac{x^2}{2y^2} = \frac{x^2 y^{-2}}{2}$ $\quad \frac{\partial u}{\partial x} = \frac{x}{y^2}$ $\quad \frac{\partial u}{\partial y} = -\frac{x^2}{y^3}$

$$\frac{\partial z}{\partial x} = e^{y}\phi'(ye^{u})\,y e^{u}\frac{x}{y^2} = e^{y+u}\phi'(ye^{u})\frac{x}{y}$$

$$\frac{\partial z}{\partial y} = z + e^{y}\phi'(ye^{u})\left(e^{u} + ye^{u}\left(\frac{-x^2}{y^3}\right)\right)$$

$$= z + e^{y+u}\phi'(ye^{u})\left(1 - \frac{x^2}{y^2}\right)$$

$$(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y}$$

$$= (x^2-y^2)e^{y+u}\phi'(ye^{u})\frac{x}{y} + xy\left(z + e^{y+u}\phi'(ye^{u})\left(1-\frac{x^2}{y^2}\right)\right)$$

$$= e^{y+u}\phi(ye^{u})\left(\frac{x^3}{y} - xy + xy - \frac{x^3}{y}\right) + xyz$$

$$= xyz$$