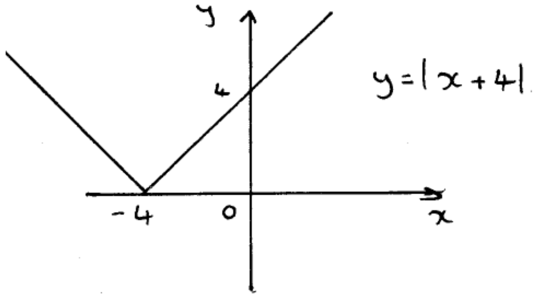
HSC Mathematics 2006

1a $e^{-0\cdot5} = 0\cdot607$ (3dp)

b $2x^2+5x-3 = (2x-1)(x+3)$

c



d $\dfrac{\sin\theta}{5} = \dfrac{\sin 33^\circ}{9}$ $\theta = \sin^{-1}\left(\dfrac{5\sin 33^\circ}{9}\right) = 18^\circ$ nearest degree

e $3-5x \le 2$ $5x \ge 1$ $x \ge \frac{1}{5}$

f $\dfrac{13}{5} + \dfrac{13}{25} + \ldots = \dfrac{\frac{13}{5}}{1-\frac{1}{5}} = \dfrac{13}{4} = 3\frac{1}{4}$

2a i $\frac{d}{dx}(x\tan x) = \tan x + x\sec^2 x$

ii $\frac{d}{dx}\left(\frac{\sin x}{x+1}\right) = \frac{(x+1)\cos x - \sin x}{(x+1)^2}$

b i $\int 1 + e^{7x}\,dx = x + \frac{1}{7}e^{7x} + C$

ii $\int_0^3 \frac{8x}{1+x^2}\,dx = 4\int_0^3 \frac{2x}{1+x^2}\,dx = \left[4\ln(1+x^2)\right]_0^3 = 4\ln 10$

c $y = \cos 2x$ $\qquad \frac{dy}{dx} = -2\sin 2x$

when $x = \frac{\pi}{6}$ $\qquad y = \frac{1}{2}$ $\qquad \frac{dy}{dx} = -\sqrt{3}$

$y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$

$6\sqrt{3}\,x + 6y - 3 - \sqrt{3}\,\pi = 0$

3a i $A(1,4)$ $B(5,-4)$ gradient of $AB = \dfrac{y_A - y_B}{x_A - x_B} = \dfrac{4 - -4}{1-5} = -2$

The equation of the straight line through A and B is

$y - 4 = -2(x-1)$ $\quad\quad 2x + y - 6 = 0$

ii At D $x = 0$ $\quad y = 6$ $\quad D(0,6)$

iii distance from (x_1, y_1) to the line $ax + by + c = 0$

is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} = \dfrac{|-3\times 2 - 1\times 1 - 6|}{\sqrt{2^2+1^2}} = \dfrac{13}{\sqrt{5}}$ units

iv Area $= \frac{1}{2} AD \times \dfrac{13}{\sqrt{5}} = \dfrac{13\sqrt{5}}{2\sqrt{5}} = 6.5$ units^2

b $\displaystyle\sum_{r=2}^{4} \frac{1}{r} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12} = 1\frac{1}{12}$

c i $u_n = a + (n-1)d$ $\quad\quad 560 - 13\times 17 = 339$

ii $S_n = \frac{n}{2}(2a + (n-1)d)$ $\quad\quad \frac{14}{2}(1120 - 13\times 17) = 6293$

iii $u_n < 60$

$560 - 17(n-1) < 60$

$517 < 17n$

$n > \dfrac{517}{17} = 30.4\ldots$

First day is 31^{st}

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4a i $\angle DBA = \frac{\pi}{6}$ $\angle DAB = \pi - 2\times\frac{\pi}{6} = \frac{2\pi}{3}$

ii $BD = \sqrt{18 - 18\cos\frac{2\pi}{3}} = 3\sqrt{3}$ m

iii Area $= \frac{1}{2}\times 9\times\sin\left(\frac{2\pi}{3}\right) + \frac{1}{2}(3\sqrt{3})^2\times\frac{5\pi}{6} = 39.2\,m^2$ (3 s.f.)

b $V = \pi\int_1^5 x^2\,dy = \pi\int_1^5 y-1\,dy = \left[\pi\left(\frac{y^2}{2} - y\right)\right]_1^5 = 8\pi$ units^3

c i $\frac{32\times 31\times 30}{64\times 63\times 62} = \frac{5}{42}$

ii $2\times\frac{5}{42} = \frac{5}{21}$

iii $1 - \frac{5}{21} = \frac{16}{21}$

5 a i $f(x) = 2x^2(3-x) = 6x^2 - 2x^3$

$f'(x) = 12x - 6x^2 = 0 \Rightarrow 6x(2-x) = 0$

$x = 0$ or $x = 2$

$y = 0$ $\quad y = 8$

$f''(x) = 12 - 12x$

$f''(0) > 0$ $\quad (0,0)$ is a local minimum

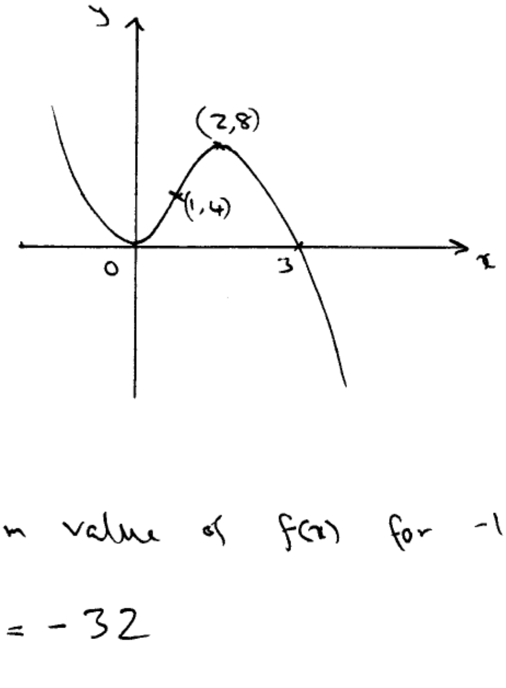
$f''(2) < 0$ $\quad (2,8)$ is a local maximum.

ii $f''(x) =$ when $x = 1$, $y = 4$

$f''(0.9) > 0$ $\quad f''(1.1) < 0$

$(1,4)$ point of inflection.

iii



iv minimum value of $f(x)$ for $-1 \leq x \leq 4$ is $f(4)$.

$f(4) = -32$

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6 a i $\angle BCA = \angle CAD$ alternate angles

$\angle CAD = \angle BAC$ (AC bisects $\angle$BAD)

$\therefore \angle BAC = \angle BCA$

ii $\angle BAC = \angle BCA$ as above

$\angle PBA = \angle PBC$ (BD bisects $\angle ABC$)

$\angle BPA = \angle BPC$ (angles of triangle ..)

BP is common to both triangles

$\therefore \Delta ABP \equiv \Delta CBP$ (ASA)

iii $\angle CBD = \angle BDA$ alternate angles

$\angle PAD = \angle PAB$ bisector

$\angle DPA = \angle APB$ angles of triangle

AP is common to Δ BAP and Δ PAD

$\therefore \Delta BAP \equiv \Delta PAD$

$\angle DPC = \angle BPA$ opposite angles

$DP = PB$ and $CP = PA$ congruence of triangles

$\Delta DPC \equiv \Delta BPA$ SAS

$\therefore DC = AB$

$AB = BC = CD = DA$ $\therefore$ ABCD is a rhombus.

6b i 450

ii $\frac{dP}{dt} = -15e^{-0.05t}$ $\left.\frac{dP}{dt}\right|_{t=10} = -9.097\ldots$

about

Population decreasing at a rate of 9 birds per year.

iii 150

iv $150 + 300e^{-0.05t} < 200$

$$e^{-0.05t} < \frac{1}{6}$$

$$-0.05t < \ln\frac{1}{6}$$

$$t > \frac{\ln\frac{1}{6}}{-0.05} = 35.835\ldots \text{ years.}$$

Species will be classified as endangered after about 36 years.

7a $x^2 - 3x + 1 = 0$

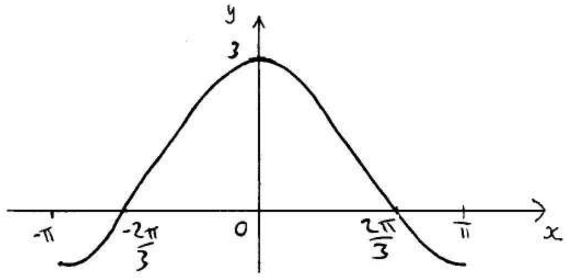
i $\alpha\beta = 1$

ii $\beta = \frac{1}{\alpha}$ $\qquad \alpha + \frac{1}{\alpha} = \alpha + \beta = 3$

b $f(x) = 1 + 2\cos x$

i $f(x) = 0 \Rightarrow x = \cos^{-1}\left(-\frac{1}{2}\right) \qquad x = \frac{2\pi}{3}$

ii



iii $$\text{Area} = \int_{-\frac{\pi}{2}}^{\frac{2\pi}{3}} (1 + 2\cos x)\,dx = \left[x + 2\sin x\right]_{-\frac{\pi}{2}}^{\frac{2\pi}{3}}$$

$$= \frac{2\pi}{3} + \sqrt{3} - \left(-\frac{\pi}{2} - 2\right) = \frac{7\pi}{6} + 2 + \sqrt{3} \text{ units}^2$$

c i $(K-2)^2 - 64 = K^2 - 4K - 60$

ii $(K-2)^2 - 64 < 0$

$(K-2)^2 < 64$

$-8 < K - 2 < 8$

$-6 < K < 10$

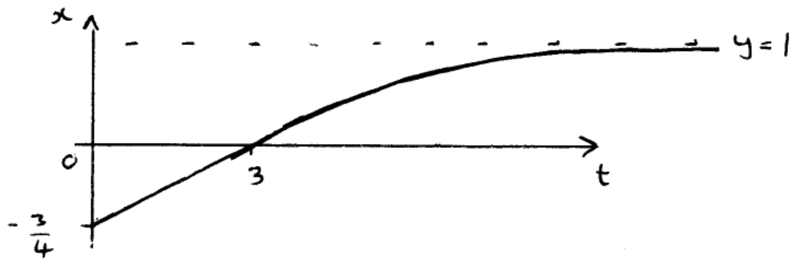
8a i $\quad x(0) = 1 - \frac{7}{4} = -\frac{3}{4}$ metres

ii $\quad \frac{dx}{dt} = \frac{7}{(t+4)^2} \qquad$ when $x = 0 \quad t+4 = 7 \quad t = 3$

$$\frac{dx}{dt} = \frac{1}{7} \text{ m/s}$$

iii $\quad a = \frac{d^2x}{dt^2} = \frac{-14}{(t+4)^3} < 0 \quad (\text{for } t > -4)$

iv



b i $\quad M = \dfrac{200000 \times 1.006^{300}}{\left(\dfrac{1 - 1.006^{300}}{1 - 1.006}\right)} = 1439.177..$

$$= \$1439.18$$

ii $\quad 2800\left(\dfrac{1.006^n - 1}{0.006}\right) = 200000 \times 1.006^n$

$$1.006^n - 1 = \frac{3}{7} \times 1.006^n$$

$$1.006^n = \frac{7}{4}$$

$$n = \frac{\log\left(\frac{7}{4}\right)}{\log(1.006)} = 93.54..$$

94 payments are required.

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9a $12y = x^2 - 6x - 3$

$12y = (x-3)^2 - 12$

$12(y+1) = (x-3)^2$ compare to $X^2 = 4 \times 3Y$ which has focus $(0, 3)$

$12y = x^2 - 6x - 3$ has focus $(3, 2)$

9bi when $t = 0$ $\frac{dv}{dt} = 120$

when $\frac{dv}{dt} = 240$

$t^2 - 26t + 120 = 0$
$(t-6)(t-20) = 0$
$t = 6$ or $t = 20$

ii $V = \int (120 + 26t - t^2)\,dt = 120t + 13t^2 - \frac{1}{3}t^3 + C$

but $V(0) = 0$ so $V(t) = 120t + 13t^2 - \frac{1}{3}t^3$

iii $V(30) = 6300$ water lost $= 1500 + 6300 - 7000$
$= 800$ Litres

9c $(x-a)^2 + r^2 = a^2$
$r^2 = 2ax - x^2$
$V = \frac{1}{3}\pi(2ax - x^2)x$
$V = \frac{1}{3}\pi(2ax^2 - x^3)$

$\frac{dV}{dx} = \frac{1}{3}\pi(4ax - 3x^2) = 0 \Rightarrow x(4a - 3x) = 0$
$\Rightarrow x = 0$ or $x = \frac{4a}{3}$

$\frac{d^2V}{dx^2} = \frac{1}{3}\pi(4a - 6x) < 0$ when $x = \frac{4a}{3}$

so max. volume when $x = \frac{4a}{3}$

10a $\displaystyle\int_{0.5}^{1.5} (\log_e x)^3 \, dx \approx \frac{1}{6}\left((\log_e \tfrac{1}{2})^3 + (\log_e \tfrac{3}{2})^3\right) = -0.04439..$

≈ -0.044

bi $QL^2 = KL^2 - KQ^2$
$= (6+x)^2 - (6-x)^2$
$= 24x$

ii Let $\angle QKL = \alpha$ then $\angle QLK = 90 - \alpha$ (angles of Δ)
$\angle NLM = 180 - \angle MLK - (90 - \alpha) = \alpha$
$\angle NML = 90 - \alpha$
$\angle NML = \angle QLK$
$\angle LNM = \angle KQL$
$\angle MLN = \angle LKQ$
The triangles are similar AA

$\dfrac{y}{MN} = \dfrac{KL}{QL}$ $\dfrac{y}{12} = \dfrac{6+x}{\sqrt{24x}}$

$y = \dfrac{12(6+x)}{2\sqrt{6}\sqrt{x}}$ $y = \dfrac{\sqrt{6}(6+x)}{\sqrt{x}}$

iii Area $= \frac{1}{2}\, y\,(6+x) = \dfrac{\sqrt{6}\,(6+x)^2}{2\sqrt{x}}$

iv $\sqrt{6}\,(6+x) \leq 13\sqrt{x}$ $6(6+x)^2 \leq 169x$
$6x^2 - 97x + 216 \leq 0$ $2\frac{2}{3} \leq x \leq 13\frac{1}{2}$ but $x \leq 6$

$\sqrt{6}\,(6+x) \geq 12\sqrt{x}$ $6(6+x)^2 \geq 144x$ $(6+x)^2 \geq 24x$
$x^2 - 12x + 36 \geq 0$ $(x-6)^2 \geq 0$
$2\frac{2}{3} \leq x \leq 6$

10v $\dfrac{dA}{dx} = \dfrac{2\sqrt{x}\,2\sqrt{6}\,(6+x) - \sqrt{6}\,(6+x)^2\,x^{-\frac{1}{2}}}{4x}$

$= \dfrac{4\sqrt{6}\,x(6+x) - \sqrt{6}(6+x)^2}{4x^{3/2}}$

$= \dfrac{\sqrt{6}(6+x)(3x-6)}{4x^{3/2}}$

> 0 when $x > 2$

So $A(x)$ is increasing for $2\frac{2}{3} \leq x \leq 6$

Minimum of A is therefore $A(2\frac{2}{3}) = 56\frac{1}{3}$ cm^2

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