7.

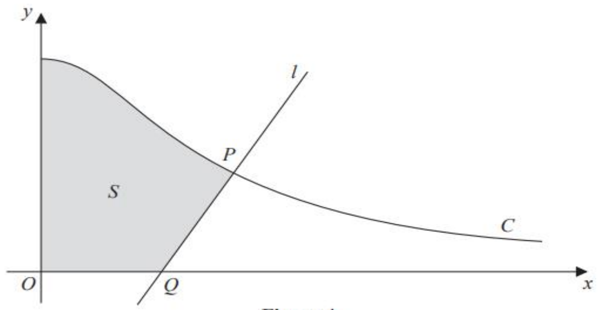

Figure 4

Figure 4 shows a sketch of part of the curve C with parametric equations

$$x = 3\tan\theta, \quad y = 4\cos^2\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point P lies on C and has coordinates (3, 2).

The line l is the normal to C at P. The normal cuts the x-axis at the point Q.

(a) Find the x coordinate of the point Q.

(6)

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the x-axis, the y-axis and the line l. This shaded region is rotated 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact value of the volume of the solid of revolution, giving your answer in the form $p\pi + q\pi^2$, where p and q are rational numbers to be determined.

[*You may use the formula* $V = \frac{1}{3}\pi r^2 h$ *for the volume of a cone.*]

(9)

$$\frac{dx}{d\theta} = 3\sec^2\theta \qquad \frac{dy}{d\theta} = -8\cos\theta\sin\theta \qquad dx = 3\sec^2\theta d\theta$$

$$\frac{dy}{dx} = -\frac{8}{3}\cos^3\theta\sin\theta$$

At P $3 = 3\tan\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ so $\frac{dy}{dx} = -\frac{8}{3}\left(\cos\frac{\pi}{4}\right)^3 \sin\frac{\pi}{4} = -\frac{2}{3}$

The gradient of the normal is $\frac{3}{2}$ and the equation of the normal is $y - 2 = \frac{3}{2}(x - 3)$.

At $Q, y = 0$ so $-2 = \frac{3}{2}(x-3) \Rightarrow -\frac{4}{3} + 3 = x \Rightarrow x = \frac{5}{3}$

The required volume is $\pi\int_0^3 y^2 dx$ −volume of cone.

When $x = 0, \theta = 0$ and when $x = 3, \ \theta = \frac{\pi}{4}$

For the cone $h = 3 - \frac{5}{3} = \frac{4}{3}$ and $r = 2$. The volume of the cone is $\frac{1}{3} \times \pi \times 2^2 \times \frac{4}{3} = \frac{16\pi}{9}$

$$V = \pi\int_0^{\frac{\pi}{4}} (4\cos^2\theta)^2 3\sec^2\theta d\theta - \frac{16\pi}{9} = 48\pi\int_0^{\frac{\pi}{4}} \cos^2\theta \ \mathrm{d}\theta - \frac{16\pi}{9}$$

$$= 24\pi\int_0^{\frac{\pi}{4}} (\cos 2\theta + 1)d\theta - \frac{16\pi}{9} = 24\pi\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{4}} - \frac{16\pi}{9} = 24\pi\left(\frac{1}{2} + \frac{\pi}{4}\right) - \frac{16\pi}{9}$$

$$= 12\pi + 6\pi^2 - \frac{16\pi}{9} = \frac{92}{9}\pi + 6\pi^2$$