

If $x = \sec \theta + \tan \theta$ show that $\frac{1}{x} = \sec \theta - \tan \theta$.

$$\begin{aligned}\frac{1}{x} &= \frac{1}{\sec \theta + \tan \theta} \\ &= \frac{\cos^2 \theta}{\cos \theta(1 + \sin \theta)} \\ &= \frac{1 - \sin^2 \theta}{\cos \theta(1 + \sin \theta)} \\ &= \frac{1 - \sin \theta}{\cos \theta} \\ &= \sec \theta - \tan \theta\end{aligned}$$

Show also that $x^2 + \frac{1}{x^2} + 2 = 4 \sec^2 \theta$.

$$\begin{aligned}x^2 + \frac{1}{x^2} + 2 &= (\sec \theta + \tan \theta)^2 + (\sec \theta - \tan \theta)^2 + 2 \\ &= \sec^2 \theta + \tan^2 \theta + 2 \sec \theta \tan \theta + \sec^2 \theta + \tan^2 \theta - 2 \sec \theta \tan \theta + 2 \\ &= 2 \sec^2 \theta + 2 \tan^2 \theta + 2 \\ &= 4 \sec^2 \theta\end{aligned}$$