# Further Pure 1 Conic Sections

The hyperbola $H$ has parametric equations $x = a\tan t$ and $y = b\sec t$.

The tangent to $H$ at the point $(a\tan t, b\sec t)$ intersects the coordinate axes at $A$ and $B$.

Find an equation for the locus of the midpoint of $AB$.

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$$\frac{dx}{dt} = a\sec^2 t \qquad \frac{dy}{dt} = b\sec t\tan t \qquad \frac{dy}{dx} = \frac{b\tan t}{a\sec t} = \frac{b}{a}\sin t$$

An equation of the tangent is

$$y - b\sec t = \frac{b}{a}\sin t\,(x - a\tan t)$$

When $x = 0$, $y = b(-\sin t\tan t + \sec t) = b\left(\frac{-\sin^2 t + 1}{\cos t}\right) = b\cos t$

When $y = 0$, $-a\sec t = x\sin t - a\sin t\tan t \Rightarrow$

$$x = a\frac{\sin t\tan t - \sec t}{\sin t} = a\left(\frac{\sin^2 t - 1}{\cos t\sin t}\right) = -a\cot t$$

The locus of the midpoint of $AB$ has parametric equations

$x = -\frac{a}{2}\cot t$ and $y = \frac{b}{2}\cos t$

$$\cot t = -\frac{2x}{a} \qquad \cos t = \frac{2y}{b}$$

$$\tan t = -\frac{a}{2x} \qquad \sec t = \frac{b}{2y}$$

Since $\sec^2 t - \tan^2 t = 1$,

$$\frac{b^2}{4y^2} - \frac{a^2}{4x^2} = 1$$

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