Nonlinear Ordinary Differential Equations, Jordan and Smith

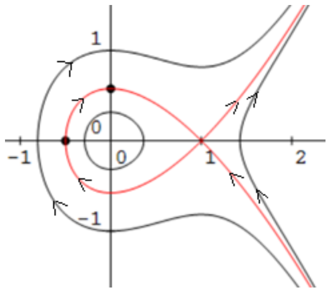
Exercise 1.2

Find the equilibrium points of the system $\ddot{x} + x - x^2 = 0$, and the general equation of the phase paths. Find the elapsed time between the points $(-\frac{1}{2}, 0)$ and $(0, \frac{1}{\sqrt{3}})$ on a phase path.

Letting $\dot{x} = y$, $\ddot{x} + x - x^2 = 0 \implies \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{x^2 - x}{y} \implies \frac{y^2}{2} = \frac{x^3}{3} - \frac{x^2}{2} + c.$

If a path passes through (-0.5, 0) then for this path $c = \frac{1}{6}$.

Equilibrium points occur where $\ddot{x} = 0$ and $\dot{x} = 0$. That is, at the points (0,0) and (1,0) in the phase diagram below, the constant solutions being $x(t) = 0$ and $x(t) = 1$.



Elapsed time between points A and B is $\int_{t=T_A}^{t=T_B} dt = \int_{t=T_A}^{t=T_B} \frac{dt}{dx} dx = \int_{x=X_A}^{x=X_B} \frac{dx}{\dot{x}} = \int_{x=X_A}^{x=X_B} \frac{dx}{y}$

Time taken to move between $\left(-\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{\sqrt{3}}\right)$ is given by

$$T = \int_{-0.5}^{0} \frac{dx}{\sqrt{\frac{2}{3}x^3 - x^2 + \frac{1}{3}}} = \int_{-0.5}^{0} \frac{dx}{\sqrt{\frac{2}{3}x^3 - x^2 + \frac{1}{3}}} = \sqrt{3} \int_{-0.5}^{0} \frac{dx}{\sqrt{2x^3 - 3x^2 + 1}}$$

$$= \sqrt{3} \int_{-0.5}^{0} \frac{dx}{\sqrt{(x-1)^2(2x+1)}} = -\sqrt{3} \int_{-0.5}^{0} \frac{dx}{(x-1)\sqrt{(2x+1)}}$$

Let $u = \sqrt{(2x+1)}$ so that $x = \frac{u^2 - 1}{2}$ and $dx = u du$

$$T = -2\sqrt{3} \int_{0}^{1} \frac{du}{u^2 - 3} = 2 \int_{0}^{1} \frac{\frac{1}{\sqrt{3}} du}{1 - \left(\frac{u}{\sqrt{3}}\right)^2} = 2 \tanh^{-1} \frac{1}{\sqrt{3}} = \log(2 + \sqrt{3})$$

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