Edexcel C4 June 2015

7. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time. Give your answer in years to 3 significant figures.

(3)

a)

$$\frac{2}{P(P-2)} \equiv \frac{A}{P} + \frac{B}{P-2} \Rightarrow 2 \equiv A(P-2) + BP$$

When $P = 0$ we have $2 = -2A \Rightarrow A = -1$ and when $P = 2$ we have $2 = 2B \Rightarrow B = 1$.

$$\frac{2}{P(P-2)} \equiv -\boldsymbol{\frac{1}{P}} + \boldsymbol{\frac{1}{P-2}}$$

b)

$$\int \frac{2}{P(P-2)}\, dP = \int \cos 2t \, dt \Rightarrow \int -\frac{1}{P} + \frac{1}{P-2}\, dP = \int \cos 2t\, dt \Rightarrow$$

$$-\ln P + \ln(P-2) = \frac{1}{2}\sin 2t + C \Rightarrow \ln\frac{P-2}{P} = \frac{1}{2}\sin 2t + C$$

Given that $P = 3$ when $t = 0$, $C = \ln\frac{1}{3}$.

$$\frac{P-2}{P} = e^{\frac{1}{2}\sin 2t + \ln\frac{1}{3}} \Rightarrow P - 2 = \frac{1}{3}Pe^{\frac{1}{2}\sin 2t} \Rightarrow$$

$$P\left(1 - \frac{1}{3}Pe^{\frac{1}{2}\sin 2t}\right) = 2 \Rightarrow P = \frac{2}{1 - \frac{1}{3}e^{\frac{1}{2}\sin 2t}} \Rightarrow \boldsymbol{P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}}$$

c)

$$4 = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}} \Rightarrow 3 - e^{\frac{1}{2}\sin 2t} = \frac{6}{4} \Rightarrow e^{\frac{1}{2}\sin 2t} = 3 - \frac{3}{2} \Rightarrow$$

$$\frac{1}{2}\sin 2t = \ln\frac{3}{2} \Rightarrow \sin 2t = 0.8109.. \Rightarrow 2t = \arcsin 0.8109.. \Rightarrow \boldsymbol{t = 0.473}$$