Core Pure 1 Matrices

Determine the values of the real constants a and b for which there are infinitely many solution to the simultaneous equations

$$2x + 3y + z = 6$$

$$-x + y + 2z = 7$$

$$ax + y + 4z = b$$

$$\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 2 \\ a & 1 & 4 \end{vmatrix} = a\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 0 \Rightarrow 5a - 5 + 20 = 0 \Rightarrow a = -3$$

If there are infinitely many solutions the planes form a sheaf (or are all the same plane, which is not the case here as the second equation is not a multiple of the first).

The line where the planes meet must pass through at least one of the planes $x = 0, y = 0$ or $z = 0$.

Trying $x = 0$.

$$3y + z = 6$$

$$y + 2z = 7$$

$$y + 4z = b$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and $b = 1 + 4 \times 3 = 13$

$$\underline{b = 13}$$

If necessary you could consider $y = 0$ or $z = 0$.