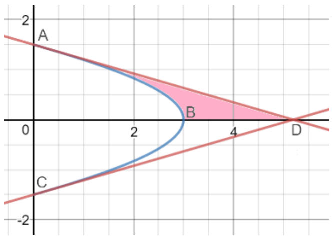
C4 Parametric Equations

The curve shown in the figure has parametric equations $x = a\cos 3t, \;\; y = a\sin t, \;\; a > 0, \quad -\frac{\pi}{6} \le t \le \frac{\pi}{6}$.



The curve meets the axes at points A, B and C as shown.

The straight lines shown are tangents to the curve at the points A and C and meet the x axis at point D. Find in terms of a

a) the equation of the tangent at A,
b) the area of the finite region between the curve, the tangent at A and the x axis, shown shaded in the figure.

Given that the total area of the finite region between the two tangents and the curve is 10 cm^2,

c) find the value of a.

Solution

At A, $x = 0$ therefore $3t = \pm\frac{\pi}{2}$ and $t = -\frac{\pi}{6}$ or $t = \frac{\pi}{6}$. For these values of t, $y = -\frac{a}{2}$ and $\frac{a}{2}$ respectively. A is the point $\left(0, \frac{a}{2}\right)$.

$\frac{dx}{dt} = -3a\sin 3t$ and $\frac{dy}{dt} = a\cos t$ therefore $\frac{dy}{dx} = -\frac{\cos t}{3\sin 3t}$. At A, $\frac{dy}{dx} = -\frac{\cos\frac{\pi}{6}}{3\sin\frac{3\pi}{6}} = -\frac{\sqrt{3}}{6}$.

The equation of the tangent at A is $y = -\frac{\sqrt{3}}{6}x + \frac{a}{2}$.

At D, $y = 0$ and $x = \frac{\frac{1}{2}a}{\frac{\sqrt{3}}{6}} = \sqrt{3}a$. The area of triangle AOD is $\frac{\sqrt{3}a \times \frac{a}{2}}{2} = \frac{\sqrt{3}a^2}{4}$.

At B, $y = 0 \Rightarrow t = 0 \Rightarrow x = a$. The area enclosed by the axes and the curve between A and B is given by $\int_0^a y\,dx$.

This is $\int_{\pi/6}^{0} a\sin t\,(-3a\sin 3t)\,dt = -3a^2\int_{\pi/6}^{0} \sin 3t \sin t\,dt$.

$\sin 3t \sin t \equiv \frac{1}{2}[(\cos 3t\cos t + \sin 3t \sin t) - (\cos 3t \cos t - \sin 3t \sin t)] \equiv \frac{1}{2}(\cos 2t - \cos 4t)$.

Alternatively, using the product sum formula, $\cos A - \cos B \equiv -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$, with $\frac{A+B}{2} = 3t$ and $\frac{A-B}{2} = t$.
$A + B = 6t, \;\; A - B = 2t \Rightarrow A = 4t$ and $B = 2t \Rightarrow \sin t \sin 3t = -\frac{1}{2}(\cos 4t - \cos 2t)$.

The integral becomes $\frac{3a^2}{2}\int_{\pi/6}^{0}(\cos 4t - \cos 2t)dt = \frac{3a^2}{2}\left[\frac{1}{4}\sin 4t - \frac{1}{2}\sin 2t\right]_{\frac{\pi}{6}}^{0}$

$= \frac{3a^2}{2}\left(\frac{1}{2}\sin\frac{\pi}{3} - \frac{1}{4}\sin\frac{2\pi}{3}\right) = \frac{3a^2}{2}\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{8}\right) = \frac{3\sqrt{3}\,a^2}{16}$.

The shaded area is the area of triangle AOD – the integral above. This is $\frac{\sqrt{3}\,a^2}{4} - \frac{3\sqrt{3}\,a^2}{16} = \frac{\sqrt{3}\,a^2}{16}$.

Given that the total area between the tangents and the curve is 10cm^2, $\frac{\sqrt{3}\,a^2}{16} = 5 \Rightarrow \sqrt{3}\,a^2 = 80 \Rightarrow a = \sqrt{\frac{80}{\sqrt{3}}} =$ 6.796 correct to 4 significant figures.

http://burymathstutor.co.uk/worked_solutions.html