C4 January 2008

8. Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 $cm^3 s^{-1}$ and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(a) Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation

$$\frac{dh}{dt} = 0.4 - k\sqrt{h}, \text{ where } k \text{ is a positive constant.}$$

(3)

When $h = 25$, water is leaking out of the hole at 400 $cm^3 s^{-1}$.

(b) Show that $k = 0.02$

(1)

(c) Separate the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h},$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20-\sqrt{h}}\, dh.$$

(2)

Using the substitution $h = (20 - x)^2$, or otherwise,

(d) find the exact value of $\int_0^{100} \frac{50}{20-\sqrt{h}}\, dh$.

(6)

(e) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

a)

$V = 4000h$ $\qquad$ $\frac{dV}{dh} = 4000$

$\frac{dV}{dt} = 1600 - c\sqrt{h}$ $\qquad$ $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1600-c\sqrt{h}}{4000} = 0.4 - k\sqrt{h}$

b)

If water is leaking out at 400 cm³/s and flowing in at 1600 cm³/s then $\frac{dV}{dt} = 1200$ cm³/s.

$\frac{dh}{dt} = \frac{\frac{dv}{dt}}{4000} = \frac{1200}{4000} = 0.3$ $\qquad$ $0.3 = 0.4 - k\sqrt{25} \Rightarrow 5k = 0.1 \Rightarrow k = 0.02$

c)

$$\int \frac{dh}{0.4 - 0.02\sqrt{h}} = \int dt \Rightarrow \int dt = \int \frac{50dh}{20 - \sqrt{h}}$$

d)

$t = \int_0^{100} \frac{50dh}{20-\sqrt{h}}$ $\qquad$ $h = (20-x)^2 \Rightarrow \frac{dh}{dx} = -40 + 2x \Rightarrow dh = (2x - 40)dx$

$$t = \int_{20}^{10} \frac{50\,(2x-40)}{20-(20-x)} dx = \int_{20}^{10} \left(100 - \frac{2000}{x}\right) dx$$

$$= (100 \times 10 - 2000 \ln 10) - (100 \times 20 - 2000 \ln 20) = 2000 \ln 2 - 1000$$

e)

$2000 \ln 2 - 1000 \approx 386.3$ seconds ≈ 6 minutes 26 seconds