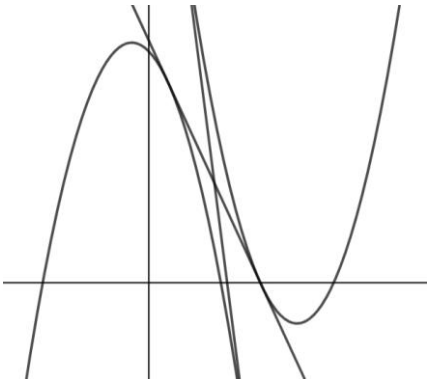
An A level Maths Question

Consider the parabolas with equations $y = -x^2 - x + 7$ and $y = x^2 - 9x + 19$.



Find the point of intersection of the two straight lines that are tangents to both parabolas.

Find the equations of the tangents.

The required point is the centre of a rotation that maps one parabola onto the other.

This is the midpoint of the line segment between the turning points.

The line of symmetry of a parabola with equation $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$ so the turning points are $\left(-\frac{1}{2}, -\left(-\frac{1}{2}\right)^2 + \frac{1}{2} + 7\right)$ and $\left(\frac{9}{2}, \left(\frac{9}{2}\right)^2 - \frac{81}{2} + 19\right)$ and the midpoint is $\left(\frac{-\frac{1}{2}+\frac{9}{2}}{2}, \frac{7-1}{2}\right)$.

The tangents both pass through the point $(2,3)$.

For the second parabola $\frac{dy}{dx} = 2x - 9$

At the point $(a, a^2 - 9a + 19)$ where a tangent meets this parabola the gradient is $2a - 9$.

$$2a - 9 = \frac{a^2 - 9a + 19 - 3}{a - 2} \Rightarrow$$

$$a^2 - 9a + 16 = (2a - 9)(a - 2) \Rightarrow$$

$$a^2 - 4a + 2 = 0 \Rightarrow a = 2 \pm \sqrt{2}$$

The gradients are $2\left(2 \pm \sqrt{2}\right) - 9 = -5 \pm 2\sqrt{2}$

The equations of the tangents are $y - 3 = \left(-5 \pm 2\sqrt{2}\right)(x - 2)$.