An A level Mathematics Question

Let $f(x) = \frac{1}{x-1} + 2$ and let $g(x) = ae^{-x} + b$.

Given that the graphs of $y = f(x)$ and $y = g(x)$ have an asymptote in common and that $g'(1) = -e$ find the values of $f'(x)$ such that $f'(x) = g'(x)$.

The equation of the common asymptote is $y = 2$.

$g'(x) = -ae^{-x}$ therefore $g'(1) = -\frac{a}{e} = -e$ and so $a = e^2$.

Where the gradients are equal $f'(x) = -\frac{1}{(x-1)^2} = -e^{2-x}$. It follows that $(x-1)^{-2} = e^{2-x}$.

By inspection $x = 2$ is a solution and $f'(2) = -1$.

To find the other solutions you may use Lambert's W function which is defined such that $W(xe^x) = x$.

$$(x-1)^2 = e^{x-2} \Rightarrow$$

$$x - 1 = \pm e^{\frac{1}{2}x-1} \Rightarrow$$

$$(x-1)e^{1-\frac{1}{2}x} = \pm 1 \Rightarrow$$

$$\left(-\frac{1}{2}x + \frac{1}{2}\right)e^{1-\frac{1}{2}x} = \pm\frac{1}{2} \Rightarrow$$

$$\left(-\frac{1}{2}x + \frac{1}{2}\right)e^{-\frac{1}{2}x+\frac{1}{2}} = \pm\frac{1}{2\sqrt{e}} \Rightarrow$$

$$-\frac{1}{2}x + \frac{1}{2} = W\left(\pm\frac{1}{2\sqrt{e}}\right) \Rightarrow$$

$$x = 1 - 2W\left(\pm\frac{1}{2\sqrt{e}}\right)$$

Now all you need is a calculator with the W function and you can find that

$x = 0.5223..$ or $x = 4.5128..$

and the required gradients are approximately -4.383 and -0.081.

Bury Maths Tutor