A Volume of Revolution Problem

The parametric equations $x = 4 + 2\sin t$ and $y = 6 + \cos t$, for $0 \le t < 2\pi$, define an ellipse.

Find the volume of the solid generated when the ellipse is rotated 2π radians about the x axis.

$$V = \pi \int_0^{2\pi} y^2 \frac{dx}{dt} dt$$

$$\frac{dx}{dt} = 2\cos t$$

$$V = 2\pi \int_0^{2\pi} \cos t\,(6 + \cos t)^2\, dt$$

$$= 2\pi \int_0^{2\pi} (36\cos t + 12\cos^2 t + \cos^3 t)\, dt$$

Integrals of odd powers of $\cos t$, $\cos 2t$ (or $\sin t$ etc.) between 0 and 2π evaluate to 0 and so

$$V = 2\pi \int_0^{2\pi} 12\cos^2 t\ dt.$$

$$\cos^2 t = \frac{(\cos 2t + 1)}{2}$$

$$V = 12\pi \int_0^{2\pi} (\cos 2t + 1)\, dt$$

$$V = 12\pi \int_0^{2\pi} 1\, dt = 24\pi^2$$

The same result can be obtained from the equation of the curve in Cartesian form.

$$V = \pi \int_2^6 \left(\left(6 + \sqrt{1 - \left(\frac{x-4}{2}\right)^2} \right)^2 - \left(6 - \sqrt{1 - \left(\frac{x-4}{2}\right)^2} \right)^2 \right) dx$$

$$= 24\pi \int_2^6 \sqrt{1 - \left(\frac{x-4}{2}\right)^2}\, dx$$

$$24\pi \int_2^6 \sqrt{1 - \left(\frac{x-4}{2}\right)^2}\, dx = \cdots = 24\pi^2.$$

Another method is to make use of Pappus' theorem which gives the volume as the area enclosed by the curve (which does not cross the x axis) multiplied by the distance travelled by the centroid. The ellipse has an area of 2π and the centroid travels 12π, on a circular path with radius 6 and so the volume is $24\pi^2$.